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New Oscillation Conditions for Second Order Half-Linear Advanced Difference Equations

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Abstract
This paper aims to establish adequate conditions that are intended for the oscillation of every solution of the second order advanced type half-linear difference equations with noncanonical form. Initially, we derive a sufficient condition that ensures all solutions of the studied equation are either oscillatory or tending to zero. Secondly, we obtain a criteria for the oscillation of all solutions of the studied equation. These criteria are obtained by using Riccati transformation and summation averaging method. The results established in this paper in essence complement, extend and enhance the existing outcomes recorded in the literature. The improvement of our main results are illustrated through three examples.

Keywords- Asymptotic behavior, Half-linear difference equation, Oscillation, Second order.

1. Introduction
The second order half-linear difference equations of the semblance

$$\Delta(a_n(\Delta y_n)^{\alpha}) + q_n y^{\alpha}_{\sigma(n)} = 0, \quad n \geq n_0, \tag{1}$$

where $\alpha > 0$, whose solutions oscillate is studied subject to the conditions:

(i) $\{a_n\}$ is a positive real sequence and $\{\sigma(n)\}$ is such that $\sigma(n) > n+1$ and $\Delta\left(\sigma\left(n\right)\right) \geq 0$ for all $n \geq n_0$;

(ii) $\{q_n\}$ is a nonnegative real sequence and it is not vanishing identically for many values of $n \geq n_0$.

A nontrivial real valued sequence $\{y_n\}$ satisfying (1) for all $n \geq n_0$ is called a solution of (1) and regarding only those solutions that exist for $n \geq N \geq n_0$ satisfying

$$\sup\{|x_n| : n \geq N_1\} > 0 \quad \text{for any} \quad N_1 \geq N.$$

A solution of (1) is oscillatory if it is neither eventually negative nor eventually positive, else termed as nonoscillatory. If all solutions of (1) oscillate, then it is regarded as oscillatory.
Adhering to Trench (1973), (1) is canonical if

$$\sum_{t=n_0}^{n-1} a_t^{-\frac{1}{\alpha}} \to \infty \quad \text{as} \quad n \to \infty. \tag{2}$$

Conversely, (1) is in noncanonical form if

$$\sum_{t=n_0}^{\infty} a_t^{-\frac{1}{\alpha}} < \infty. \tag{3}$$

Difference equations with advanced argument finds applications in the progression of growth rate which not only rely on the current, but extends to the near future. Inducing an advanced argument persuades the subsequent actions that are immediately accessible and helpful for decision making. Economic crisis and dynamics of population for instance are the phenomenal complications, contemplated to manifest (Elsgolts and Norkin, 1973; Agarwal, 2000).

A very great attention were received recently for the oscillatory and asymptotic behavior of solutions of difference equation (Agarwal, 2000; Agarwal et al., 2005). Though there was much investigation done on delay difference equations, meager studies were devoted to equations having advanced arguments (Zhang and Cheng, 1995; Zhang and Zhang, 1999; Thandapani et al., 2001; Ping and Han, 2003; Agarwal et al., 2005; Ocalan and Akin, 2007; Arul and Ayyapan, 2013; Selvarangam et al., 2016; Wu et al., 2016).

Ping and Han (2003) considered the following equation

$$\Delta(a_n(\Delta y_n)) + p_n y_n + q_n y_{\sigma(n)} = 0, \quad n \geq n_0, \tag{4}$$

and obtained some sufficient conditions for the oscillation of all solutions of equation (4) when it is in canonical form.

Ocalan and Akin (2007) considered the following equation

$$\Delta y_n + p_n y_{n+k} = 0, \quad n \geq n_0, \tag{5}$$

and established several sufficient conditions for the oscillation of all solutions of equation (5)

Zhang and Li (1998) considered the following equation

$$\Delta(a_n(\Delta y_n)) + q_n y_{\sigma(n)} = 0, \quad n \geq n_0, \tag{6}$$

and derived several oscilltion criteria through Riccati equation when equation (6) is in canonical form.

Arul and Ayyappan (2013) considered the following equation

$$\Delta(a_n\Delta(y_n + p_n y_{n+k})) + q_n y_{n+l} + v_n y^{\alpha}{}_{n+m} = 0, \quad n \ge n_0, \tag{7}$$

and obtained conditions for the oscillation of all solutions of equation (7) when it is in canonical form.

From the above review of literature one can see that all the oscillation results established for the advanced type difference equations are linear and in canonical form. Therefore in this paper we obtain oscillation criteria for second order advanced difference equation with noncanonical form since such equations include Euler-type difference equations as a special case.

2. Main Results

Henceforth assume that (3) holds. Define

$$A(n) = \sum_{s=n}^{\infty} a_s^{\frac{-1}{\alpha}}.$$

Let us consider only the positive solutions of (1), for if $\{y_n\}$ satisfies (1), then $\{-y_n\}$ also does.

Lemma 1 *Assume that*

$$\sum_{n=n_0}^{\infty} q_n = \infty. \tag{8}$$

Further, assume that (1) has a positive solution $\{y_n\}$ for all $n \ge n_1 \ge n_0$. Then

$$y_n > 0, \quad \Delta y_n < 0, \quad \Delta(a_n(\Delta y_n)^{\alpha}) \le 0, \tag{9}$$

for $n \ge n_1$. Moreover $\left\{\dfrac{y_n}{A(n)}\right\}$ is nondecreasing for $n \ge n_1$.

Proof: Let $\{y_n\}$ be a positive solution of (1) for $n \ge n_1$. From (1), we have

$$\Delta(a_n(\Delta y_n)^{\alpha}) = -q_n y^{\alpha}_{\sigma(n)} \le 0.$$

This implies that, $\{\Delta y_n\}$ is either positive or negative eventually. To the contrary suppose that (8) holds, and there is a $n_2 \ge n_1$ such that $\Delta y_n > 0$ for $n \ge n_2$. Define

$$w_n = \frac{a_n(\Delta y_n)^{\alpha}}{y^{\alpha}_{\sigma(n)}} > 0, \qquad n \ge n_2,$$

and therefore $w_n > 0$ and

$$\Delta w_n = -q_n - \frac{a_{n+1}(\Delta y_{n+1})^{\alpha}}{y^{\alpha}_{\sigma(n+1)} y^{\alpha}_{\sigma(n)}} \Delta y^{\alpha}_{\sigma(n)} \le -q_n. \tag{10}$$

Summing (10) from n_2 to $n-1$, results in

$$w_n \le w_{n_2} - \sum_{s=n_2}^{n-1} q_s.$$

In view of (8), it is evident that the last inequality contradicts the positivity of $\{w_n\}$. This justifies that $\Delta y_n > 0$ is not possible implying that $\{y_n\}$ satisfies (9) for all $n \ge n_1$.

From the monotonicity of $a_n^{\frac{1}{\alpha}} \Delta y_n$ that for $l \ge n$,

$$y_n \ge -\sum_{s=n}^{l} a_s^{\frac{-1}{\alpha}} a_s^{1/\alpha} \Delta y_s \ge -a_n^{\frac{1}{\alpha}} \Delta y_n \sum_{s=n}^{l} a_s^{\frac{-1}{\alpha}}.$$

Letting $l \to \infty$, we see that

$$y_n \ge -a_n^{\frac{1}{\alpha}} \Delta y_n A(n). \tag{11}$$

Now we have from (11)

$$\Delta\left(\frac{y_n}{A(n)}\right) = \frac{a_n^{1/\alpha} A(n)\Delta y_n + y_n}{a_n^{\frac{1}{\alpha}} A(n) A(n+1)} \ge 0. \tag{12}$$

Hence $\left\{\dfrac{y_n}{A(n)}\right\}$ is nondecreasing, which completes the proof.

Theorem 2 *Suppose that*

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{a_n} \sum_{s=n_0}^{n-1} q_s \right)^{\frac{1}{\alpha}} = \infty. \tag{13}$$

Furthermore, if (1) has a positive solution $\{y_n\}$ for all $n \ge n_1 \ge n_0$, then $\{y_n\}$ satisfies (9) for all $n \ge n_1$ and

$$\lim_{n\to\infty} y_n = 0. \tag{14}$$

Proof: Let $\{y_n\}$ be a positive solution of (1) for $n \ge n_1$. From (13) and (3), the unboundedness of $\sum_{s=n_0}^{n-1} q_s$ is evident, implying (8) holds. By Lemma 1, $\{y_n\}$ satisfies (9) for $n \ge n_1$.

Since $\{y_n\}$ is positive decreasing, there is $M \ge 0$ such that $\lim_{n\to\infty} y_n = M$. Assume $M > 0$. Then there is an integer $n_2 \ge n_1$ such that

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$$-\Delta(a_n(\Delta y_n)^\alpha) = q_n y^\alpha_{\sigma(n)} \geq M^\alpha q_n, \quad n \geq n_2.$$

Taking summation from n_2 to $n-1$, in the last inequality, leads to

$$-a_n(\Delta y_n)^\alpha \geq a_{n_2}(\Delta y_{n_2})^\alpha + M^\alpha \sum_{s=n_2}^{n-1} q_s,$$

that is,

$$-\Delta y_n \geq M\left(\frac{1}{a_n}\sum_{s=n_2}^{n-1} q_s\right)^{\frac{1}{\alpha}}. \tag{15}$$

Summing (15) from n_2 to $n-1$, we obtain

$$y_n \leq y_{n_2} - M_2 \sum_{s=n_2}^{n-1}\left(\frac{1}{a_s}\sum_{t=n_2}^{s-1} q_t\right)^{\frac{1}{\alpha}}.$$

In the last inequality using (13) implies that $y_n \to -\infty$, as $n \to \infty$ which is a contradiction to the positivity of y_n. Thus $M = 0$, and the proof is complete.

Theorem 3 *If*

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{a_n}\sum_{s=n_0}^{n-1} A^\alpha(\sigma(s))q_s\right)^{\frac{1}{\alpha}} = \infty, \tag{16}$$

then (1) oscillates.

Proof: Assume the contrary that $\{y_n\}$ is a solution of (1) such that $y_n > 0$, $y_{\sigma(n)} > 0$ for all $n \geq n_1 \geq n_0$. Note that (8) is essential for (16) to hold. The function

$$\sum_{s=n_0}^{n-1} A^\alpha(\sigma(s))q_s$$

is unbounded due to (3) and $\Delta A(n) < 0$, (8) must hold. Then, by Lemma 2.1, $\{y_n\}$ satisfies (9) for all $n \geq n_1$. It follows from $\left\{\frac{y_n}{A(n)}\right\}$ is nondecreasing there is a constant $M > 0$ and $n_2 \geq n_1$ such that $y_n \geq MA(n)$ for $n \geq n_2$. Substituting this inequality into (1), we see that

$$-\Delta(a_n(\Delta y_n)^\alpha) \geq M^\alpha q_n A^\alpha(\sigma(n)). \tag{17}$$

Summing (17) from n_2 to $n-1$, we have

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$$-a_n(\Delta y_n)^\alpha \geq M^\alpha \sum_{s=n_2}^{n-1} q_n A^\alpha(\sigma(s)),$$

that is,

$$-\Delta y_n \geq \frac{M}{a_n^{\frac{1}{\alpha}}}\left(\sum_{s=n_2}^{n-1} q_s A^\alpha(\sigma(s))\right)^{\frac{1}{\alpha}}.$$

Again taking summation from n_2 to $n-1$ and taking (16) into account, we get

$$y_{n_2} \leq y_n - \sum_{s=n_2}^{n-1} \frac{M}{a_s^{\frac{1}{\alpha}}}\left(\sum_{t=n_2}^{s-1} q_t A^\alpha(\sigma(t))\right)^{\frac{1}{\alpha}} \to -\infty, \text{ as } n \to \infty,$$

This contradiction completes the proof.

In the next theorem, we derive the oscillation criteria for the case when (16) does not satisfy.

Theorem 4 *If*

$$\limsup_{n\to\infty} A^\alpha(\sigma(n)) \sum_{s=n_1}^{n-1} q_s > 1, \tag{18}$$

for any $n_1 \geq n_0$, then (1) oscillates.

Proof: Assume the contrary that $\{y_n\}$ is a solution of (1) such that $y_n > 0$, $y_{\sigma(n)} > 0$ for $n \geq n_1 \geq n_0$. First note that (18) along with (3) imply (8). Then by Lemma 1, $\{y_n\}$ satisfies (9) for all $n \geq n_1$. Summing (1) from n_1 to $n-1$ and applying the fact that $\{y_n\}$ is decreasing, we have

$$-a_n(\Delta y_n)^\alpha = -a_{n_1}(\Delta y_{n_1})^\alpha + \sum_{s=n_1}^{n-1} q_s y_{\sigma(s)}^\alpha$$

$$\geq y_{\sigma(n)}^\alpha \sum_{s=n_1}^{n-1} q_s$$

Using Lemma 1, we obtain (11), and together with the above inequality leads to

$$-a_n(\Delta y_n)^\alpha \geq -a_{\sigma(n)}(\Delta y_{\sigma(n)})^\alpha A^\alpha(\sigma(n)) \sum_{s=n_1}^{n-1} q_s$$

$$\geq -a_n(\Delta y_n)^\alpha A^\alpha(\sigma(n)) \sum_{s=n_1}^{n-1} q_s,$$

and hence we obtain

$$\limsup_{n\to\infty} A^\alpha(\sigma(n)) \sum_{s=n_1}^{n-1} q_s \leq 1.$$

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This contradicts (18) and the proof is complete.

Theorem 5 *Suppose that condition (8) holds and if there is a positive solution* $\{\rho_n\}$ *such that*

$$\limsup_{n\to\infty}\left\{\frac{A^{\alpha}(n)}{\rho_n}\sum_{s=N}^{n-1}\left(\rho_s q_s\left(\frac{A(\sigma(s))}{A(s)}\right)^{\alpha}-\frac{a_s}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho_s)^{\alpha+1}}{\rho_s^{\alpha}}\left(\frac{A(s)}{A(s+1)}\right)^{\alpha(\alpha+1)}\right)\right\}>1 \quad (19)$$

for any $N\geq n_0$, then (1) oscillates.

Proof: Assume the contrary that $\{y_n\}$ is a solution of (1) such that $y_n>0$, and $y_{\sigma(n)}>0$, for all $n\geq n_1\geq n_0$. By Lemma 2.1, $\{y_n\}$ satisfies (9) for $n\geq n_1$. Define

$$v_n=\rho_n\left(\frac{a_n(\Delta y_n)^{\alpha}}{y_n^{\alpha}}+\frac{1}{A^{\alpha}(n)}\right),\quad n\geq n_1. \quad (20)$$

By virtue of (11), we see that $v_n\geq 0$ for all $n\geq n_1$. From (20) we have

$$\Delta v_n=\frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1}+\rho_n\frac{\Delta(a_n(\Delta y_n)^{\alpha})}{y_n^{\alpha}}-\frac{\rho_n a_{n+1}(\Delta y_{n+1})^{\alpha}}{y_n^{\alpha}y_{n+1}^{\alpha}}\Delta y_n^{\alpha}+\rho_n\Delta\left(\frac{1}{A^{\alpha}(n)}\right). \quad (21)$$

From (1) and (12), we have

$$\Delta(a_n(\Delta y_n)^{\alpha})\leq-\left(\frac{A(\sigma(n))}{A(n)}\right)^{\alpha}q_n y_n^{\alpha},\quad n\geq n_2\geq n_1. \quad (22)$$

Using (22) in (21) one obtains

$$\Delta v_n\leq\frac{\Delta\rho_n}{\rho_{n+1}}v_{n+1}-\rho_n q_n\left(\frac{A(\sigma(n))}{A(n)}\right)^{\alpha}-\rho_n\left(\frac{v_{n+1}}{\rho_{n+1}}-\frac{1}{A^{\alpha}(n+1)}\right)\frac{\Delta y_n^{\alpha}}{y_n^{\alpha}}$$
$$+\rho_n\Delta\left(\frac{1}{A^{\alpha}(n)}\right),\quad n\geq n_2 \quad (23)$$

By Mean Value Theorem,

$$\Delta y_n^{\alpha}=\alpha t^{\alpha-1}\Delta y_n,$$

where $y_{n+1}<t<y_n$. Since $\Delta y_n<0$, we see that

$$\Delta y_n^{\alpha}\leq\alpha\frac{y_{n+1}^{\alpha}}{y_n}\Delta y_n. \quad (24)$$

From (23) and (24), we obtain

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$$\Delta v_n \le -\rho_n q_n \left(\frac{A(\sigma(n))}{A(n)}\right)^{\alpha} + \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1} - \alpha\rho_n \left(\frac{v_{n+1}}{\rho_{n+1}} - \frac{1}{A^{\alpha}(n+1)}\right) \frac{y_{n+1}^{\alpha}}{y_n^{\alpha+1}} \Delta y_n + \rho_n \Delta\left(\frac{1}{A^{\alpha}(n)}\right)$$

$$\le -\rho_n q_n \left(\frac{A(\sigma(n))}{A(n)}\right)^{\alpha} + \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1}$$

$$-\alpha\rho_n \left(\frac{v_{n+1}}{\rho_{n+1}} - \frac{1}{A^{\alpha}(n+1)}\right) \frac{A^{\alpha+1}(n+1)}{A^{\alpha+1}(n)} \frac{\Delta y_n}{y_{n+1}^{\alpha}} + \rho_n \Delta\left(\frac{1}{A^{\alpha}(n)}\right),$$

where we have used $\left\{\frac{y_n}{A(n)}\right\}$ is nondecreasing. Again using $a_n^{\frac{1}{\alpha}} \Delta y_n$ is decreasing and then from (20), we obtain

$$\Delta v_n \le -\rho_n q_n \left(\frac{A(\sigma(n))}{A(n)}\right)^{\alpha} + \frac{\Delta\rho_n}{\rho_{n+1}} v_{n+1}$$

$$-\frac{\alpha\rho_n}{a_n^{\frac{1}{\alpha}}} \left(\frac{A(n+1)}{A(n)}\right)^{\alpha+1} \left(\frac{v_{n+1}}{\rho_{n+1}} - \frac{1}{A^{\alpha}(n+1)}\right)^{1+\frac{1}{\alpha}} + \rho_n \Delta\left(\frac{1}{A^{\alpha}(n)}\right).$$

Let $A = \frac{\Delta\rho_n}{\rho_{n+1}}$, $B = \frac{\alpha\rho_n}{a_n^{\frac{1}{\alpha}} \rho_{n+1}^{1+\frac{1}{\alpha}}} \left(\frac{A(n+1)}{A(n)}\right)^{\alpha+1}$, $C = \frac{\rho_{n+1}}{A^{\alpha}(n+1)}$ and using Lemma 6 of Wu et al. (2016), we get

$$\Delta v_n \le -\rho_n q_n \left(\frac{A(\sigma(n))}{A(n)}\right)^{\alpha} + \frac{a_n}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho_n)^{\alpha+1}}{\rho_n^{\alpha}} \left(\frac{A(n)}{A(n+1)}\right)^{\alpha(\alpha+1)} + \Delta\left(\frac{\rho_n}{A^{\alpha}(n)}\right), \quad n \ge n_2. \tag{25}$$

Summing (25) from n_2 to $n-1$, we obtain

$$\sum_{s=n_2}^{n-1} \left(\rho_s q_s \left(\frac{A(\sigma(s))}{A(s)}\right)^{\alpha} - \frac{a_s}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho_s)^{\alpha+1}}{\rho_s^{\alpha}} \left(\frac{A(s)}{A(s+1)}\right)^{\alpha(\alpha+1)}\right) - \frac{\rho_n}{A^{\alpha}(n)} + \frac{\rho_{n_2}}{A^{\alpha}(n_2)} \le v_{n_2} - v_n.$$

Using the definition of v_n in the above inequality leads to

$$\sum_{s=n_2}^{n-1} \left(\rho_s q_s \left(\frac{A(\sigma(s))}{A(s)}\right)^{\alpha} - \frac{a_s}{(\alpha+1)^{\alpha+1}} \frac{(\Delta\rho_s)^{\alpha+1}}{\rho_s^{\alpha}} \left(\frac{A(s)}{A(s+1)}\right)^{\alpha(\alpha+1)}\right)$$

$$\le \rho_{n_2} \frac{a_{n_2} (\Delta y_{n_2})^{\alpha}}{y_{n_2}^{\alpha}} - \rho_n \frac{a_n (\Delta y_n)^{\alpha}}{y_n^{\alpha}}. \tag{26}$$

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On the other hand, from (11), it follows that

$$\frac{-\rho_n}{A^\alpha(n)} \le \rho_n \frac{a_n(\Delta y_n)^\alpha}{y_n^\alpha} \le 0. \tag{27}$$

Combining (26) and (27), we get

$$\sum_{s=n_2}^{n-1}\left(\rho_s q_s\left(\frac{A(\sigma(s))}{A(s)}\right)^\alpha - \frac{a_s}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho_s)^{\alpha+1}}{\rho_s^\alpha}\left(\frac{A(s)}{A(s+1)}\right)^{\alpha(\alpha+1)}\right)$$
$$\le \frac{\rho_n}{A^\alpha(n)} + \rho_{n_2}\frac{a_{n_2}(\Delta y_{n_2})^\alpha}{y_{n_2}^\alpha}. \tag{28}$$

Multiplying (28) by $\dfrac{A^\alpha(n)}{\rho_n}$ and then taking lim sup as $n \to \infty$, we get

$$\limsup_{n\to\infty}\frac{A^\alpha(n)}{\rho_n}\sum_{s=n_2}^{n-1}\left(\rho_s q_s\left(\frac{A(\sigma(s))}{A(s)}\right)^\alpha - \frac{a_s}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho_s)^{\alpha+1}}{\rho_s^\alpha}\left(\frac{A(s)}{A(s+1)}\right)^{\alpha(\alpha+1)}\right) \le 1,$$ which contradicts (18). This completes the proof.

By taking $\rho_n = A^\alpha(n)$, $\rho_n = A(n)$, and $\rho_n = 1$ following corollaries are consequent from Theorem 2.5, respectively.

Corollary 6 *Assume that (4) holds. If*

$$\limsup_{n\to\infty}\sum_{s=N}^{n-1}\left(q_s A^\alpha(\sigma(s)) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{A^{\alpha^2+\alpha}(s)}{a_s^{\frac{1}{\alpha}}A^{\alpha^2+\alpha+1}(s+1)}\right) > 1, \tag{29}$$

for any $N \ge n_0$, then (1) oscillates.

Corollary 7 *Assume that (4) holds. If*

$$\limsup_{n\to\infty}\left(A^{\alpha-1}(n)\sum_{s=N}^{n-1} q_s\frac{A^\alpha(\sigma(s))}{A^{\alpha-1}(s)} - \frac{1}{(\alpha+1)^{\alpha+1}a_s^{\frac{1}{\alpha}}}\frac{A^{\alpha^2}(s)}{A^{\alpha^2+\alpha}(s+1)}\right) > 1, \tag{30}$$

for any $N \ge n_0$, then (1) is oscillates.

Corollary 8 *Assume that (4) holds. If*

$$\limsup_{n\to\infty}\left(A^\alpha(n)\sum_{s=N}^{n-1} q_s\left(\frac{A(\sigma(s))}{A(s)}\right)^\alpha\right) > 1, \tag{31}$$

for any $N \ge n_0$, then (1) is oscillates.

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3. Examples

This section illustrates the effectiveness of our outcomes having the subsequent difference equations.

Example 1. *Consider the second order advanced type difference equation*

$$\Delta(n^{\frac{5}{3}}(n+1)^{\frac{5}{3}}(\Delta y_n)^{\frac{5}{3}})+n^2 y_{2n}^{\frac{5}{3}}=0, \quad n\geq 1, \tag{32}$$

Here $a_n=n^{\frac{5}{3}}(n+1)^{\frac{5}{3}}$, $q_n=n^2$, $\sigma(n)=2n$, and $\alpha=\dfrac{5}{3}$, and

$$A(n)=\sum_{s=n}^{\infty}\frac{1}{s(s+1)}=\frac{1}{n}.$$

Clearly the conditions (3) and (8) are satisfied, and the condition (13) becomes

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^{\frac{5}{3}}(n+1)^{\frac{5}{3}}}\sum_{s=1}^{n-1}s^2\right)^{\frac{3}{5}}\approx\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{5}}}=\infty.$$

Then by Theorem 5 any nonoscillatory of (32) converges to zero as $n\to\infty$.

Example 2. *Consider the following difference equation*

$$\Delta(n^{\frac{1}{3}}(n+1)^{\frac{1}{3}}(\Delta y_n)^{\frac{1}{3}})+n^{\frac{4}{3}} y_{2n}^{\frac{1}{3}}=0, \quad n\geq 1, \tag{33}$$

Here $a_n=(n(n+1))^{\frac{1}{3}}$, $q_n=n^{\frac{4}{3}}$, $\sigma(n)=2n$, and $\alpha=\dfrac{1}{3}$. By simple calculation, we obtain

$$A(n)=\sum_{s=n}^{\infty}\frac{1}{a_n^{\frac{1}{\alpha}}}=\sum_{s=n}^{\infty}\frac{1}{s(s+1)}=\frac{1}{n}.$$

The conditions (3) and (8) are clearly satisfied and the condition (16) becomes

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left(\sum_{s=1}^{n-1}\frac{1}{(2s)^{\frac{1}{3}}}s^{\frac{4}{3}}\right)^{3}\sum_{n=1}^{\infty}\frac{n^2(n-1)^3}{2^{\frac{4}{3}}(n+1)}=\infty.$$

Then by Theorem 6, every solution of (33) is oscillatory.

Example 3. *Consider the following difference equation*

$$\Delta(n(n+1)\Delta y_n)+\lambda y_{2n}=0, \quad n\geq 1, \lambda>0. \tag{34}$$

Here $a_n=n(n+1)$, $q_n=\lambda$, $\sigma(n)=2n$, $\alpha=1$, and

$$A(n) = \sum_{s=n}^{\infty} \frac{1}{s(s+1)} = \frac{1}{n}.$$

The conditions (3) and (8) are clearly satisfied and the condition (18) becomes

$$\lim_{n\to\infty} \sup \frac{1}{2n} \sum_{s=1}^{n-1} \lambda = \lim_{n\to\infty} \sup \frac{\lambda(n-1)}{2n} = \frac{\lambda}{2}.$$

If $\lambda > 2$, then by Theorem 7, every solution of equation (34) is oscillatory.

4. Conclusion

The results presented in this paper are new and of high degree of generality. The results obtained in the literature (Zhang and Li, 1998; Ping and Han, 2003; Arul and Ayyappan, 2013) are applicable only when the studied equation is linear and canonical, but the results presented in this paper are applicable to linear and half-linear equations with noncanonical forms. Therefore the results provided in this paper complement, extend and enhance the existing outcomes recorded in the literature. Further three examples are provided to dwell upon the importance of our main results.

It might also be interesting to extend the results of this paper to higher order advanced type difference equation

$$\Delta(a_n(\Delta^{m-1}y_n)^{\alpha}) + q_n y_{\sigma(n)}^{\beta} = 0, \quad n \geq n_0,$$

where $m \geq 2$ is an even integer and α and β are ratio of odd positive integers.

Conflict of Interest

The authors confirm that this article contents have no conflict of interest.

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